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Quantum Mechanics, Knot Theory, and Quantum Doubles

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In previous papers we have described quantum mechanics as a matrix symplectic geometry and showed the existence of a braiding and Hopf algebra structure behind our lattice quantum phase space. The first aim of this work is to give the defining commutation relations of the quantum Weyl–Schwinger–Heisenberg group associated with our $\mathfrak{R}$-matrix solution. The second aim is to describe the knot formalism at work behind the matrix quantum mechanics. In this context, the quantum mechanics of a particle–antiparticle system ($p\bar{p}$) moving in the quantum phase space is viewed as a quantum double.

1. INTRODUCTION

In Djemai (1996) we studied the *quantum mechanical symplectic geometry*. We showed that the *noncommutative* character of this geometry leads to a description of *quantization* in the Weyl–Schwinger procedure as a star deformation of the algebra of classical observables.

Notice that the Schwinger basis may also be used in other contexts. For instance, this technique has been used in the study of the *discrete* torus membrane (Floratos, 1989a), in the construction of special representations of the $GL_q(2)$ quantum group (Floratos, 1989b), and in the Manin plane description of quantum mechanics (Floratos, 1990). The Z_2-grading of the algebra

$$[\gamma_{\mathbf{n}}, \gamma_{\mathbf{m}}] = 2i \sin[\alpha_2(\mathbf{m}, \mathbf{n})]\ \gamma_{\mathbf{m}+\mathbf{n}}$$

and its *fermionic* counterpart

$$\{\gamma_{\mathbf{n}}, \gamma_{\mathbf{m}}\} = 2 \cos[\alpha_2(\mathbf{m}, \mathbf{n})]\ \gamma_{\mathbf{m}+\mathbf{n}}$$

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have been also used in the $N \to \infty$ case as a supersymmetric example of Kac–Moody algebra (Fairlie *et al.*, 1989).

In other respects, the Fourier analysis, on the one hand, is the appropriate framework to describe the quantization and, on the other hand, leads naturally to Hopf algebras (Kirillov, 1976).

In fact, there are many ways to approach Hopf algebras (Sweedler, 1969; Abe, 1980). Moreover, these new algebraic structures have given rise to the notion of *quantum groups* (Drinfel'd, 1986), and their relation to the *Yang–Baxter equation* (Yang, 1967; Baxter, 1972, 1982) and to the *braid group* (Birman, 1974) is well established (Alvarez-Gaumé *et al.*, 1989a,b; Sierra, 1989; Jimbo, 1989; Majid, 1990a; Kosmann-Schwarzbach, 1990).

In general, a *quantum group* may be defined as a *noncommutative* and *noncocommutative Hopf algebra*. There are several approaches to quantum groups in the literature (Drinfel'd, 1986; Jimbo, 1985; Woronowicz, 1986, 1987a,b; Manin, 1988; Faddeev *et al.*, 1989). Another approach (Dubois-Violette and Launer, 1990) consists in introducing the quantum group, in analogy with the definition of classical groups, as a transformation group that preserves a nondegenerate bilinear form.

Furthermore, *knot theory* (Rolfsen, 1976; Burde and Zieschang, 1986; Kauffman, 1987a, 1991; Lickorish, 1988; Birman, 1991; Bruschi, 1993) is also another way to approach quantum groups. Indeed, it is well known that the Jones polynomial (Jones, 1985) leads naturally to the notion of quantum groups and that quantum groups give rise to *link invariants* via solutions of the Yang–Baxter equation (Akutsu *et al.*, 1989; Turaev, 1988; Kauffman, 1990a).

In Djemai (1995) we studied the existence of a braiding behind the *lattice quantum phase space* introduced in Djemai (1996) and presented two different nontrivial solutions to the resulting Yang–Baxter equation. The first one is obtained by using the fact that the quantum group that preserves a nondegenerate bilinear form (in our case, it is our *matrix symplectic structure* $\Omega_{\mathbf{ab}}$) is a Hopf algebra defined by a multiplicative matrix with an $\mathcal{R}$-matrix given by (Dubois-Violette and Launer, 1990)

$$\mathcal{R}^{\mathbf{ij}}_{\mathbf{ab}} = \delta^{\mathbf{i}}_{\mathbf{b}}\delta^{\mathbf{j}}_{\mathbf{a}} + \alpha\Lambda^{\mathbf{ji}}\Omega_{\mathbf{ab}}$$

The second solution is extracted from the generalized composition law between the Schwinger matrices $\gamma^{\mathbf{m}}$:

$$\gamma^{\mathbf{m}} \cdot \gamma^{\mathbf{n}} = \exp[2i\alpha_2(\mathbf{n}, \mathbf{m})]\ \gamma^{\mathbf{n}} \cdot \gamma^{\mathbf{m}} = \sum_{\mathbf{a},\mathbf{b}} \mathcal{B}^{\mathbf{mn}}_{\mathbf{ab}} \gamma^{\mathbf{a}} \cdot \gamma^{\mathbf{b}}$$

where the commutation coefficients $\mathcal{B}^{\mathbf{mn}}_{\mathbf{ab}}$ are given by

$$\mathcal{B}^{\mathbf{mn}}_{\mathbf{ab}} = \delta^{\mathbf{m}+\mathbf{n}}_{\mathbf{a}+\mathbf{b}} \exp\{i[\alpha_2(\mathbf{a}, \mathbf{b}) - \alpha_2(\mathbf{m}, \mathbf{n})]\}$$

obey the braid equation

$$\mathcal{B}_{12}\mathcal{B}_{23}\mathcal{B}_{12} = \mathcal{B}_{23}\mathcal{B}_{12}\mathcal{B}_{23}$$

leading to the following $\mathcal{R}$-matrix solution:

$$\mathcal{R}_{\mathbf{ab}}^{\mathbf{mn}} = \mathcal{B}_{\mathbf{ab}}^{\mathbf{nm}}$$

In this case, the $\mathcal{R}$-matrices are of $(N^2 - 1)^2 \times (N^2 - 1)^2$ type. For instance, for $N = 2$, $\mathcal{R}$ is a 9×9 matrix (Djemai, 1995).

The main aim of this work is twofold. First, we discuss the construction of the matrix quantum group corresponding to the above $\mathcal{R}$-matrix. Second, we show that a quantum double $D(H)$ (Drinfel'd, 1986), where H is a Hopf algebra, may give rise to a universal link invariant via a solution to the quantum Yang–Baxter equation. One application of this quantum double (QD) construction consists in considering the Hopf algebra H to be a matrix algebra $M_N(\mathrm{C})$ generated by Schwinger matrices $\gamma_{\mathbf{m}}$, and the Hopf algebra H^*, dual to H, to be the matrix algebra generated by the $\gamma^{\mathbf{m}}$. The continuous case is also considered. It enables us to describe the quantum mechanics (QM) of a pair $p\bar{p}$ as a QD.

This work is organized as follows. In Section 2, we show how knot theory can lead naturally via the Kauffman bracket to the notion of quantum groups. In Section 3, we construct the quantum group associated with our $\mathcal{R}$-matrix solution for $N = 2$ given in Djemai (1995). The results appear to be somewhat trivial, but we expect that this treatment will become more interesting from the case $N = 3$ forward. We also give the algebraic relations for $N = 2$ using a generalization of the classical notion of exponentiation in the context of the $SU_q(2)$ quantum group. In Section 4, we develop our Hopf algebra structure to construct a quantum double which reproduces a knot theory. In this context, we introduce a new diagrammatic notation which is more general than the Kauffman one (Kauffman, 1990a). The continuous case is also studied and some physical interpretations are proposed. Finally, Section 5 is devoted to concluding remarks.

2. KNOT THEORY, YANG–BAXTER EQUATION, AND QUANTUM GROUPS

The *Reidemeister moves* (Reidemeister, 1932) of types I, II, and III (see Fig. 1) can be performed on a *link diagram*. A *link diagram* is a locally four-valent plane graph with extra structure at the vertices in the form of *crossings* (see Fig. 2).

A *link* is a collection of circles embedded in R^3 such that its projection onto a plane gives rise to a *diagram* for this link. A *knot* is a link with one component, i.e., a single circle embedded in R^3.

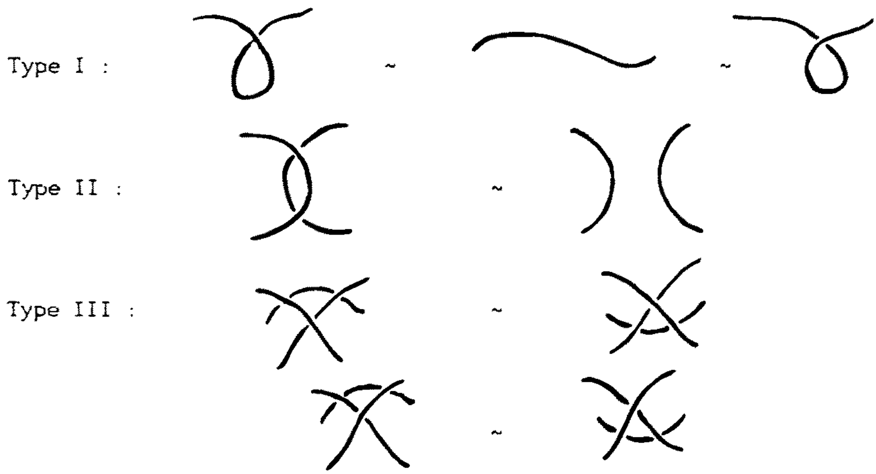

Fig. 1. The Reidemeister moves.

Two links are said to be *ambient isotopic* if there is a continuous time-parameter family of embeddings starting with one link and ending with the other one. The theory of knots and links is the theory of link embeddings under the equivalence relation of ambient isotopy. This theory is generated by the Reidemeister moves through the statement that *two links (or knots) are ambient isotopic if and only if their diagrams (planar projections) are related by a finite sequence of Reidemeister moves.*

However, there is no constructive way to do this or even to establish if this can be done.

If one restricts this equivalence relation to that generated by Reidemeister moves of types II and III only, the two links are said to be *regularly isotopic.* The example illustrated in Fig. 3 shows that opposite curls cancel. So, this (topological) property is suitable for the case of *framed links.* A *framed link* is a link such that each component has a continuous normal vector field. This leads us to think about embeddings of *bands* rather than circles.

Figure 4 gives an example of a band which no longer has invariance under Reidemeister move type I. Although the bands illustrated in Fig. 5 are isotopic, their diagrams are not regularly isotopic, since their Whitney degrees (Whitney, 1937) are different (see Fig. 6). A useful invariant of regular

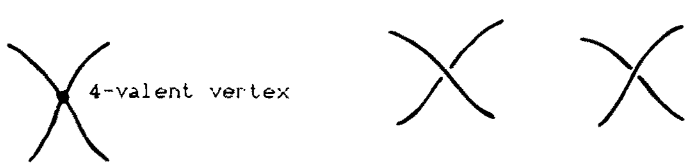

Fig. 2. Various crossings.

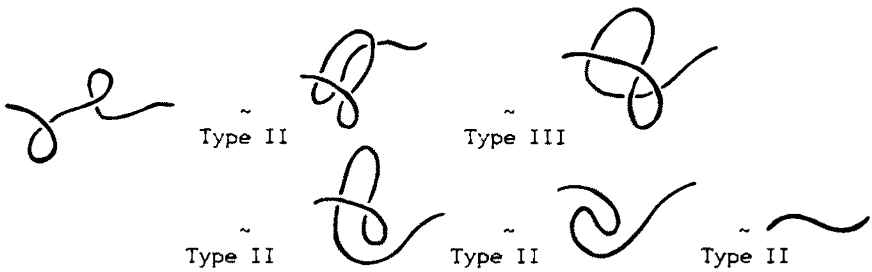

Fig. 3. Regular isotopy: opposite curls cancel.

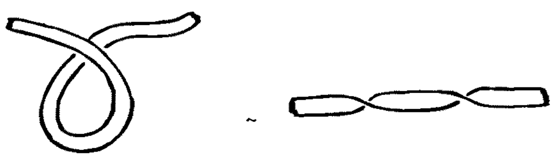

Fig. 4. Framed link: no invariance under the type I move.

isotopy is the *sum of the crossing signs*, i.e., the so-called *writhe* $w(K)$ of a link diagram K (see Fig. 7).

Remarkable progress has been made by the introduction of *link (or knot) invariants* (Alexander, 1928), namely quantities that do not change while deforming the original diagram through the Reidemeister moves. But we do not have a complete set of these invariants, so link (or knot) invariants are not the definitive answer to this problem.

Here we present a brief description of the bracket model (Kauffman, 1987b, 1990a; Akutsu *et al.*, 1989; Turaev, 1988) of the Jones polynomial (Jones, 1985) by associating a well-defined polynomial in three variables $\langle K\rangle(\mathcal{A},\mathcal{B},\mathcal{d})$ to an unoriented link K. The polynomial $\langle K\rangle$ is completely determined by the two formulas in Fig. 8.

The first formula asserts that the polynomial for a given diagram is obtained as an additive combination of the polynomials for the diagrams

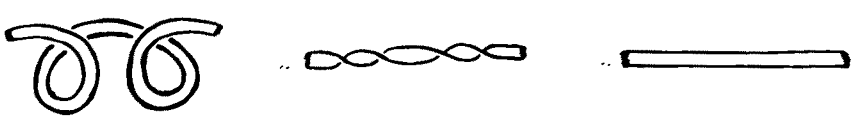

Fig. 5. Isotopic bands.

Fig. 6. No regular isotopy for the link diagrams corresponding to the above bands (see Fig. 5).

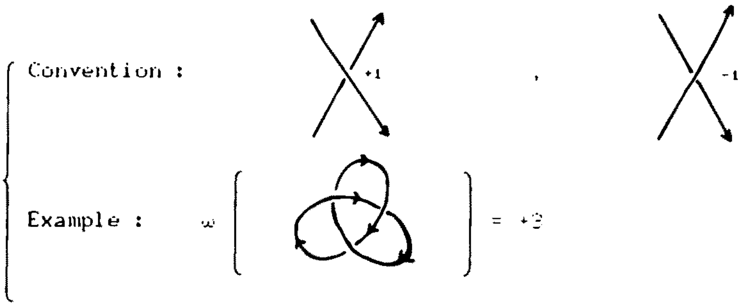

Fig. 7. The sign convention of the writhe, and an example.

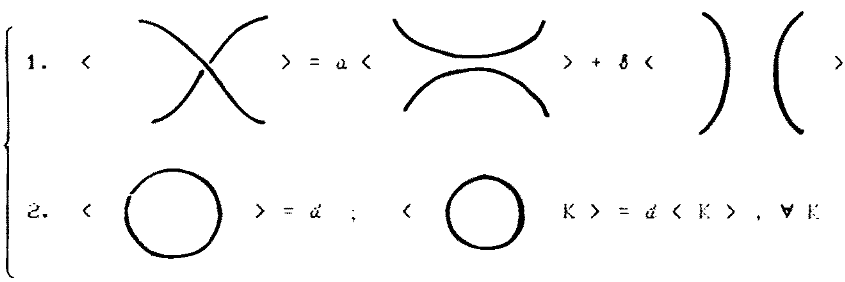

Fig. 8. Formulas 1 and 2 associating a polynomial to an unoriented link.

obtained by splicing away the given crossing in two possible ways. The second formula indicates the value "d" of a loop if it occurs isolated inside a larger diagram. Then, the value of the polynomial acquires a factor d^N from a disjoint union of N loops.

As it stands, $\langle K\rangle$ is not an invariant of any of the Reidemeister moves. However, Fig. 9 shows a formula which is a simple consequence of formulas 1 and 2 in Fig. 8. This bracket is an invariant of the Reidemeister move of type II if and only if we choose b and d such that

$$\begin{cases} b = a^{-1} \\ d = -a^{2} - a^{-2} \end{cases} \tag{1}$$

It also follows directly that $\langle K\rangle$ is invariant under the move III (see Fig. 10). Then, $\langle K\rangle$ is an invariant of regular isotopy.

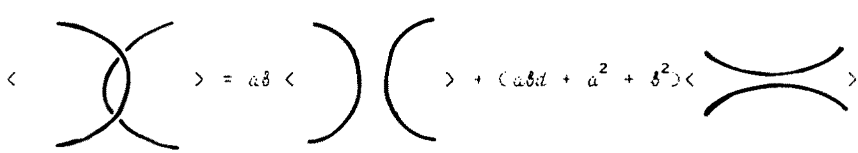

Fig. 9. The polynomial corresponding to the type II move.

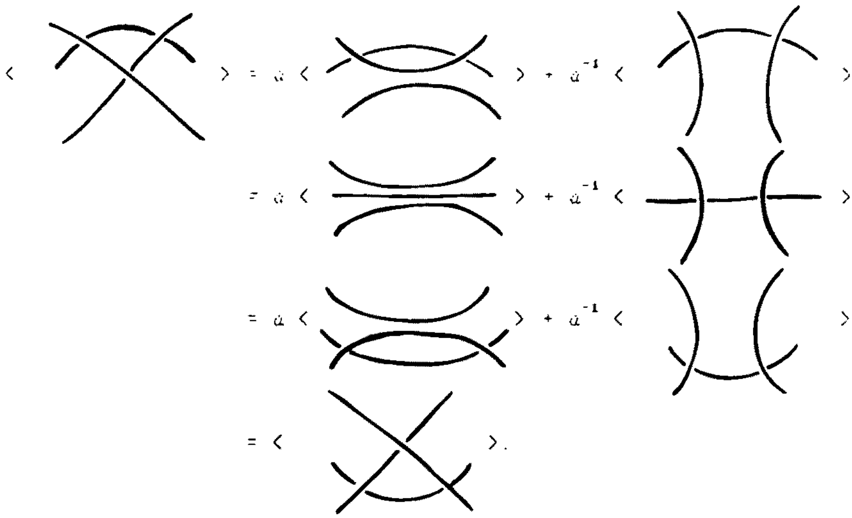

Fig. 10. The invariance corresponding to the type III move.

To obtain an invariant of ambient isotopy for oriented links, one uses

$$\mathfrak{f}_K(a) = (-a^3)^{-w(K)}\langle K\rangle/\langle O\rangle \tag{2}$$

where K is oriented, $w(K)$ is the writhe of K, and $\langle K\rangle$ is the bracket evaluated on the unoriented link underlying K. The factor $(-a^3)$ comes from the evaluation of the bracket of the move I (see Fig. 11).

Then, to any oriented link K, one can associate the following Jones polynomial (Kauffman, 1987b):

$$V_K(t) = \mathfrak{f}_K(t^{-1/4}) \tag{3}$$

Now, we will show how the Kauffman bracket polynomials can be seen as a *vacuum–vacuum amplitude* in a combinatorial version of a topological quantum field theory (Witten, 1989).

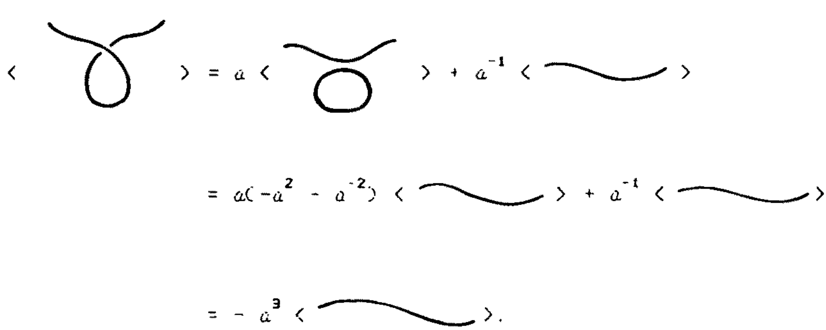

Fig. 11. The factor $(-a^3)$ coming from the polynomial of the type I move.

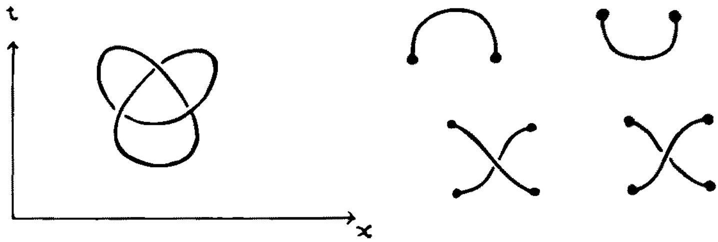

Fig. 12. A link diagram in the 1+1 space-time. Maxima, minima, and crossings.

Let us regard the projection plane as 1 + 1 space-time and position the link diagram so that it is transversal to the space levels except at the critical points corresponding to maxima, minima, and crossings (see Fig. 12).

Each minimum can be regarded as a *creation of two particles* from the vacuum, each maximum as an *annihilation*, and each crossing as an *interaction*. To each of these events one can associate a matrix whose indices go over (say) the spins of the particles and whose values are the *amplitudes* for each of these events (see Fig. 13). These amplitudes are calculated according to the principles of quantum mechanics (Feynman, 1964):

(a) If an event is decomposable into a set of steps (creations, annihilations, interactions), then the amplitude of this event is the product of the amplitudes of all the steps.

(b) If an event may occur in several disjoint alternative ways, then its amplitude is the sum of the amplitudes of all the ways.

Given a diagram K and a set of matrices $\mathcal{M}$ and $\mathcal{R}$ (see Fig. 13), one can compute the amplitude $T(K)$ for this diagram. This amplitude decomposes as a sum of the amplitudes for *configurations* of the diagram. Each configuration is an assignment of spins to the *nodes* of the diagram. Given a configuration, each matrix has a well-defined value and the amplitude of this configuration is the product of these values. Thus, the *vacuum–vacuum amplitude* $T(K)$, i.e., the amplitude for particles to be created from the vacuum, interact in the pattern of the link diagram K, and return to the vacuum, is

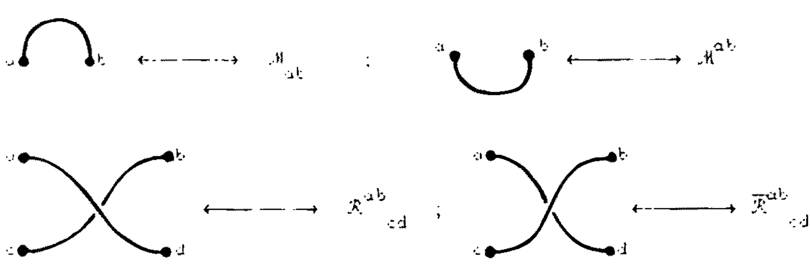

Fig. 13. Matrix quantities associated with maxima, minima, and crossings.

the sum (over all the configurations) of the product of the matrix values for each configuration.

For instance, the associated vacuum–vacuum amplitude of the link diagram presented in Fig. 14 is given by

$$T(K) = \mathcal{M}_{ab}\mathcal{M}_{cd}\mathcal{R}^{bc}_{ef}\overline{\mathcal{R}}^{ae}_{gh}\overline{\mathcal{R}}^{fd}_{ij}\mathcal{M}^{gj}\mathcal{M}^{hi} \tag{4}$$

In order for $T(K)$ to be an invariant of regular isotopy, one needs the following restrictions on the matrices, corresponding to Figs. 15a–15d, respectively:

$$1. \quad \mathcal{M}_{ai}\mathcal{M}^{ib} = \delta^b_a \tag{5a}$$

$$2. \quad \mathcal{R}^{ab}_{ij}\overline{\mathcal{R}}^{ij}_{cd} = \delta^a_c\delta^b_d \tag{5b}$$

$$3. \quad \overline{\mathcal{R}}^{ab}_{cd} = \mathcal{M}_{ci}\mathcal{R}^{ia}_{dj}\mathcal{M}^{jb} \tag{5c}$$

$$4. \quad \mathcal{R}^{ab}_{ij}\mathcal{R}^{jc}_{kf}\mathcal{R}^{ik}_{de} = \mathcal{R}^{bc}_{ij}\mathcal{R}^{ai}_{dk}\mathcal{R}^{kj}_{ef} \tag{5d}$$

Whereas the meaning of the two first conditions is clear, it is interesting to try to give a consistent interpretation for the other two.

The third condition relates $\mathcal{R}$ and $\mathcal{R}^{-1}$ via creations and annihilations. A possible physical meaning of this condition may be understood through the following example. In Fig. 16, we assume that the condition 1 holds and that parallel identity lines are interchangeable with pairs of creations and annihilations. Then, one obtains an *equivalence of spin and statistics* (Sorkin, 1988), where spin is associated to the twist of framing (curl of the diagram) and statistics corresponds to the braiding of the two lines (see Fig. 17).

Condition 4 gives us simply the Yang–Baxter equation. Then, behind a knot diagrammatic structure there is a quantum Hopf algebra structure.

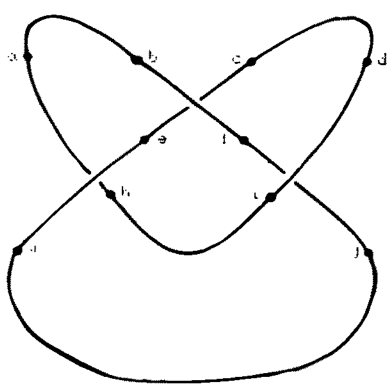

Fig. 14. A link diagram K whose associated vacuum–vacuum amplitude $T(K)$ is given by equation (4).

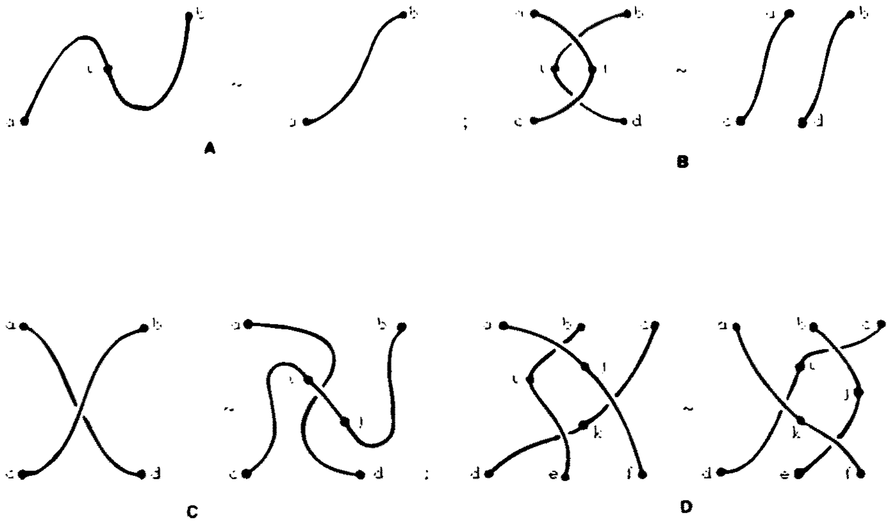

Fig. 15. Diagrammatic restrictions ensuring regular isotopy invariance for the above $T(K)$ (see Fig. 14).

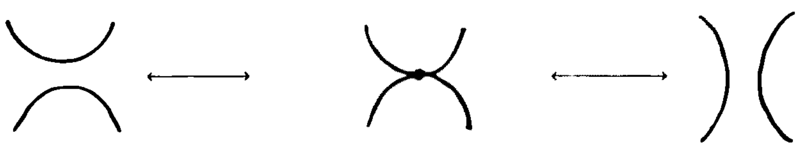

Fig. 16. Parallel lines are interchangeable with pairs of creations and annihilations.

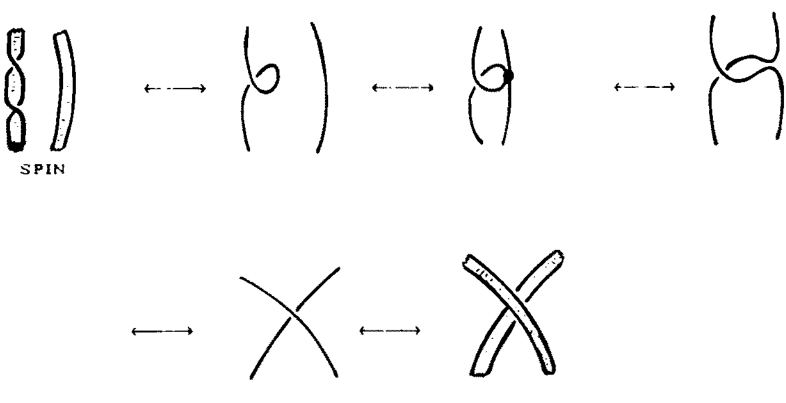

Fig. 17. Equivalence spin–statistics.

Furthermore, by simply adjusting the creation and annihilation matrices correctly, one automatically produces a model of the bracket and a solution to the Yang–Baxter equation which will coincide with the $\mathcal{R}$-matrix corresponding to $SL_q(2)$.

First, in order to model the bracket with a vacuum–vacuum amplitude, one needs to determine creation and annihilation matrices that are inverse to one another and that give a loop value $\mathcal{d} = -a^2 - a^{-2}$ [see equation (1)].

A possible solution to this problem is given by (Kauffman, 1988, 1990b)

$$\mathcal{M} = \begin{pmatrix} 0 & \sqrt{-1}\,a \\ -\sqrt{-1}\,a^{-1} & 0 \end{pmatrix} = \sqrt{-1}\,\tilde{\eta} \tag{6}$$

where the loop value $\mathcal{d}$ is correctly adjusted since it corresponds to the sum of the squares of the entries of $\mathcal{M}$, and

$$\mathcal{M}^2 = 1, \qquad \mathcal{M}_{ab}\mathcal{M}^{bc} = \delta^c_a, \qquad \mathcal{M}_{ab} = \mathcal{M}^{ab} \tag{7}$$

Fixing the choice for the creations and annihilations, there is one choice for the $\mathcal{R}$-matrix to give the following bracket (see Fig. 18):

$$\mathcal{R}^{ab}_{cd} = a\,\mathcal{M}^{ab}\mathcal{M}_{cd} + a^{-1}\delta^a_c\delta^b_d \tag{8}$$

With this choice, $T(K)$ will satisfy the defining equations of the bracket (see Fig. 8), and therefore

$$\langle K\rangle = T(K) \tag{9}$$

With the relation (7), which is not else than condition 1 [see equation (5a)], and considering $\mathcal{R}$ to be defined by equations (5), then condition 2 [see equation (5b)] follows immediately:

$$\mathcal{R}\overline{\mathcal{R}} = \mathbf{1} \tag{10}$$

Condition 3 is also satisfied, as is shown in Fig. 19, while condition 4 is proved by first checking (see Fig. 20)

$$\mathcal{R}^{ab}_{cd}\mathcal{M}^{de} = \mathcal{M}^{ad}\overline{\mathcal{R}}^{be}_{dc} \tag{11}$$

and then performing the following variation on the bracket derivation of the invariance under the type III move (see Fig. 21).

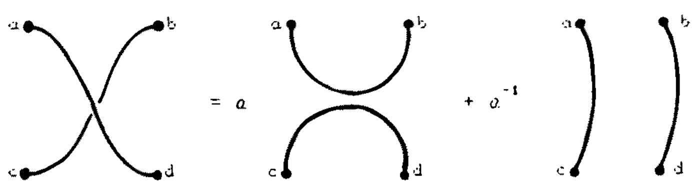

Fig. 18. The bracket corresponding to formula 1 of Fig. 8 with $\mathcal{A} = a^{-1}$ [see equations (1)].

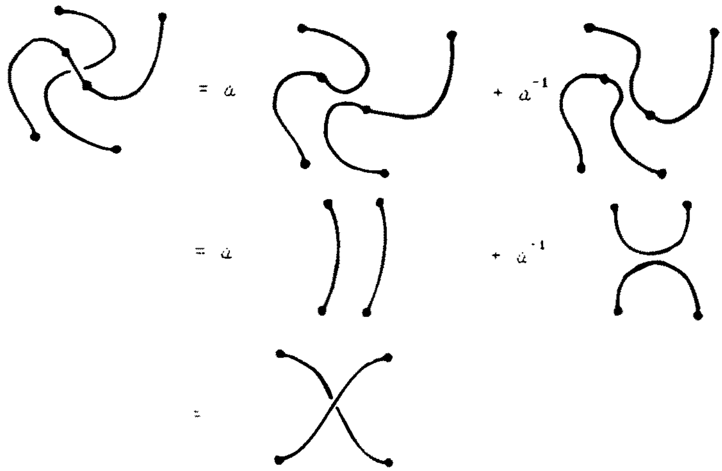

Fig. 19. An example illustrating condition 3, equation (5c).

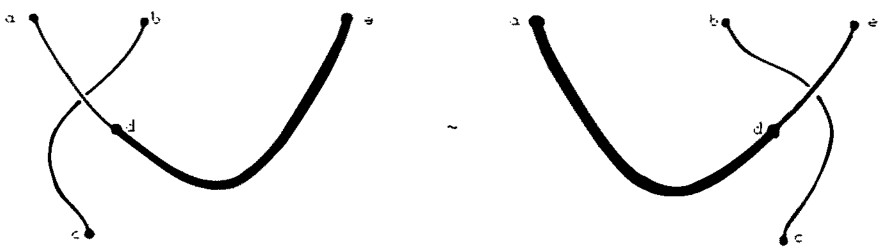

Fig. 20. An intermediate scheme corresponding to equation (11).

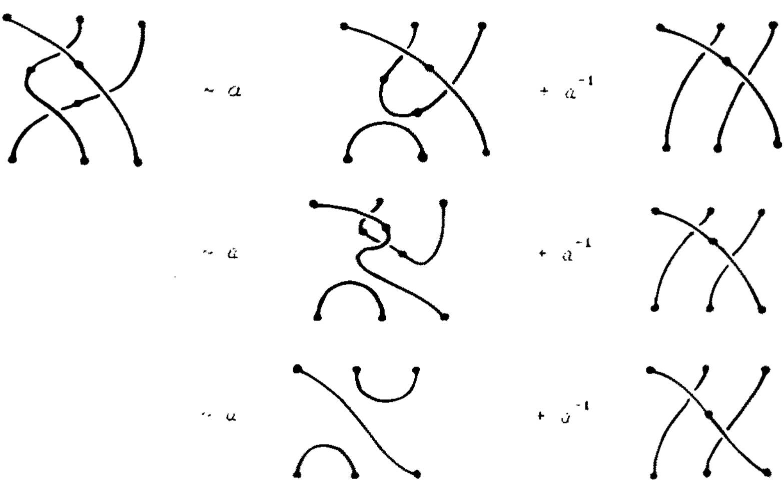

Fig. 21. Symmetrical middle point.

Then, we have shown that it is possible to produce a model of the Kauffman bracket and to give a solution to the Yang–Baxter equation by an appropriate adjustment of the matrices $\mathcal{M}$ of creation and annihilation.

In other respects, the matrix $\tilde{\eta}$ [see equation (6)] satisfies

$$\lim_{a \to 1} \tilde{\eta} \to \eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{12}$$

Let $A \in GL(2)$; then η satisfies the well-known relation

$$A \eta A^t = \mathrm{Det}(A)\, \eta \tag{13}$$

If $A \in SL(2)$, then $\mathrm{Det}(A) = 1$ and $SL(2)$ can be considered as the set of matrices A leaving invariant the bilinear form η:

$$SL(2) = \{A / A \eta A^t = \eta\} \tag{14}$$

At $a \to 1$, the bracket does not discern between under and overcrossings, and the identity illustrated in Fig. 22 corresponds directly to the Fierz identity:

$$\eta^{ab} \eta_{cd} = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c \tag{15}$$

Thus, at $a \to \pm 1$, these diagrams are interpreted as tensor diagrams for $SL(2)$-invariant quantities.

It is then natural to think about a generalization of this symmetry for the topology of link diagrams. That is, what is the *quantum group* associated to the bilinear form $\tilde{\eta}$ (Dubois-Violette and Launer, 1990)?

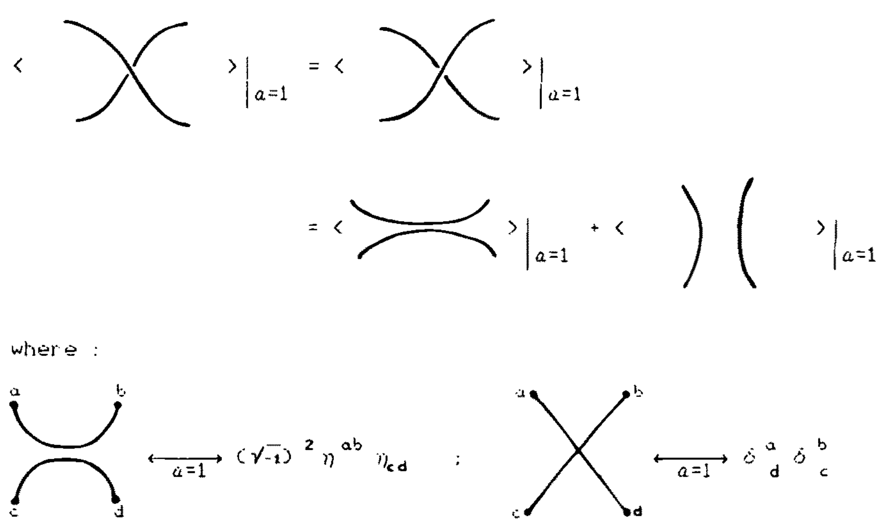

Fig. 22. Fierz identity.

Let us consider the set of matrices A:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{16}$$

with associative and possibly noncommutative entries, and ask for the following invariances:

$$A\tilde{\eta}A^t = \tilde{\eta}, \qquad A^t\tilde{\eta}A = \tilde{\eta} \tag{17}$$

It is easy to see that these conditions are equivalent to the defining relations for the quantum group $A_q = \text{Fun}_q(SL(2))$:

$$a \cdot b = qb \cdot a, \qquad a \cdot c = qc \cdot a, \qquad a \cdot d - d \cdot a = (q - q^{-1})b \cdot c$$
$$b \cdot c = c \cdot b, \qquad b \cdot d = qd \cdot b, \qquad c \cdot d = qd \cdot c \tag{18}$$

with

$$q = \sqrt{\alpha} \tag{19}$$

The coproduct of this *quantum* Hopf algebra is given by

$$\Delta\colon \quad A_q \to A_q \otimes A_q / \Delta(A^i_j) = \sum_k A^i_k \otimes A^k_j \tag{20}$$

The antipode γ and the counit ϵ are defined, respectively, by

$$\gamma(A) = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} \Leftrightarrow \begin{cases} \sum_k \gamma(A^i_k)A^k_j = \delta^i_j \\ \sum_k A^i_k\gamma(A^k_j) = \delta^i_j \end{cases} \tag{21}$$

$$\epsilon(A^i_j) = \delta^i_j \tag{22}$$

3. QUANTUM WEYL–SCHWINGER–HEISENBERG GROUP

First, let us comment on duality. For a classical compact group G, the algebra $A_1 = \text{Fun}(G)$ of *representative functions* on G carries a Hopf algebra structure. The correspondence between G and A_1 is given by Tannaka's duality theorem (see, for instance, Abe, 1980). Indeed, the group structure on G gives rise to a Hopf algebra structure on A_1: Multiplication in G defines a coproduct $\Delta\colon A_1 \to A_1 \otimes A_1$, evaluation at the identity element defines a counit $\epsilon\colon A_1 \to \mathbb{C}$, and the inversion map $g \to g^{-1}$ in G defines an antipode $\gamma\colon A_1 \to A_1$.

If G is locally compact, commutative, and compact (or discrete), then its *dual space* $\tilde{G}$ is locally compact, commutative, and compact (or discrete). This is the so-called *Pontryagin duality* (Woronowicz, 1980). For instance, the Fourier-dual spaces of the group R of real numbers is itself, of the circle S^1 is the group Z (and vice versa), and of the cyclic group Z_N is itself.

In other respects, the *quantum group* $A_q = \text{Fun}_q(G)$ is a noncommutative and noncocommutative Hopf algebra depending on a complex parameter such that $\text{Fun}_1(G) = \text{Fun}(G)$. Its *dual space* is no longer a group, but a *category* (Majid, 1990b), that of finite-dimensional unitary representations of the considered quantum group. This is the so-called *Tannaka–Krein duality* (Majid, 1990c).

Furthermore, the commutative group has a property which does not generalize in the noncommutative case. That is, the unitary irreducible representations are 1-dimensional and the tensor product of two such representations is another 1-dimensional one. Each such representation may be considered simply as a complex function $f: G \to C$ with $f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$ for any $g_1, g_2 \in G$. The tensor product is then reduced to the simple pointwise product of functions and the set of inequivalent unitary irreducible representations is then itself a group. This is the *Pontryagin duality.* In the quantum case this property is not fulfilled and one ends up with the *Tannaka–Krein duality* between quantum groups and categories.

Finally, one can associate with a classical group G another classical Hopf algebra, namely the *universal enveloping algebra* $\mathcal{U}(\mathcal{G})$ of the Lie algebra $\mathcal{G}$ of G. This Hopf algebra admits a one-parameter deformation $\mathcal{U}_q(\mathcal{G})$ (Kirillov and Reshetikhin, 1988), and it is well known that there is some *duality* between $\text{Fun}_q(G)$ and $\mathcal{U}_q(\mathcal{G})$ (Rosso, 1987; Meister and Wong, 1991).

From this discussion, we can emphasize the interesting duality property of the cyclic group Z_N, that is, it is dual to itself.

However, we know that the Heisenberg group is non-Abelian.

Making use of the Weyl–Wigner–Moyal formalism in the context of the discrete Weyl–Schwinger realization of the Heisenberg group with the prescription that *noncommutativity* is absorbed in the coefficients $a^{\mathbf{m}}$ of an operator $\mathbf{A}$ belonging to the group algebra and written in the Schwinger basis $\{\gamma_{\mathbf{m}}\}$ (Djemai, 1995, 1996)

$$\mathbf{A} = \frac{1}{N} \sum_{\mathbf{m}} a^{\mathbf{m}} \gamma_{\mathbf{m}} \tag{23}$$

then one may pass from the algebra of functions on $Z_N \otimes Z_N$ with a twisted product $*_\nu$ to the algebra of functions on $Z_N \otimes Z_N$ with the (operator) product $\circ$. This allows us to use the (Fourier)–Pontryagin duality instead of the Tannaka–Krein one, since the dual space of the commutative group $Z_N \otimes$

Z_N is itself. This trick can be used to avoid the explicit use of Hopf algebras (Aldrovandi, 1993).

In Djemai (1995) we presented a new nontrivial solution $\mathcal{R}$ to the Yang–Baxter equation. For arbitrary N, it is given by the following $(N^2 - 1)^2 \times (N^2 - 1)^2$ matrix:

$$\mathcal{R}^{\mathbf{mn}}_{\mathbf{ab}} = \delta^{\mathbf{m+n}}_{\mathbf{a+b}} \omega^{(\mathbf{a \times b + m \times n})/2} \tag{24}$$

with $\omega = \exp(i2\pi/N)$. For instance, for $N = 2$ we get (Djemai, 1995)

$$\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & \omega & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \omega^{-1} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \omega & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \omega^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \omega^{-1} & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{25}$$

In general, the RTT equations (Faddeev *et al.*, 1989)

$$\mathcal{R}T_1T_2 = T_2T_1\mathcal{R}, \qquad T_1 = B \otimes \mathbf{1}, \qquad T_2 = \mathbf{1} \otimes B \tag{26}$$

will permit us to determine the *defining commutation relations* of the *quantum group* associated with this $\mathcal{R}$-matrix. The most general one is obtained by taking the matrix B in equation (26) to be a general $(N^2 - 1) \times (N^2 - 1)$-matrix:

$$B = (B_{ij}), \qquad i, j = 1, 2, \ldots, N^2 - 1 \tag{27}$$

The simplicity of the case $N = 2$ produces, as expected, trivial results due to the fact that $\omega = -1$. It is easy to see that all the matrix elements are central and obey the following relations:

$$B_{11}B_{12} + B_{21}B_{22} + B_{31}B_{32} = 0 \tag{28a}$$

$$B_{11}B_{13} + B_{21}B_{23} + B_{31}B_{33} = 0 \tag{28b}$$

$$B_{11}B_{21} + B_{12}B_{22} + B_{13}B_{23} = 0 \tag{28c}$$

$$B_{11}B_{31} + B_{12}B_{32} + B_{13}B_{33} = 0 \tag{28d}$$

$$B_{12}B_{13} + B_{22}B_{23} + B_{32}B_{33} = 0 \tag{28e}$$

$$B_{21}B_{31} + B_{22}B_{32} + B_{23}B_{33} = 0 \tag{28f}$$

and

$$(B_{11})^2 - (B_{22})^2 = (B_{23})^2 - (B_{31})^2 = (B_{32})^2 - (B_{13})^2 \tag{29a}$$

$$(B_{22})^2 - (B_{33})^2 = (B_{13})^2 - (B_{21})^2 = (B_{31})^2 - (B_{12})^2 \tag{29b}$$

Hence, it is more interesting to investigate the case $N \geq 3$, where the deformation parameter ω is a root of unity different from -1. Moreover, the obtained Hopf algebra in this case is defined by the following operations:

$$\Delta: \quad A \to A \otimes A / \Delta(B_{ij}) = \sum_k B_{ik} \otimes B_{kj} \tag{30a}$$

$$\gamma: \quad A \to A / \gamma(B_{ij}) = (B^{-1})_{ij} \tag{30b}$$

$$\epsilon: \quad A \to \mathrm{C} / \epsilon(B_{ij}) = \delta_{ij} \tag{30c}$$

However, different quotients of the resulting *quantum group* can be considered by imposing additional (consistent) relations on the generators B_{ij}. For instance, one can choose B to be an *upper-triangular* matrix:

$$B_{ij} = 0 \qquad \text{for} \quad i < j \tag{31}$$

The case $N \geq 3$ will be treated in a future paper.

We end this section by presenting another way of looking at the situation $N = 2$. In general, it is well known that an ordinary Lie group can be locally obtained from the Lie algebra by exponentiation and that the infinitesimal generators of the Lie group can be identified with the Lie algebra. For quantum groups and quantum algebras, both of these relations are replaced by duality of Hopf algebras. However, it was shown in Vokos *et al.* (1990) and Corrigan *et al.* (1990) that the elements of a 2×2 $SU_q(2)$ matrix which is infinitesimally near the identity element satisfy simple commutation relations and when exponentiated give an element of $SU_q(2)$. Thus, consider an element of $SU_q(2)$:

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{32}$$

where the entries a, b, c, and d satisfy the relations

$$a \cdot b = qb \cdot a, \qquad a \cdot c = qc \cdot a, \qquad a \cdot d - d \cdot a = (q - q^{-1}) b \cdot c$$

$$b \cdot c = c \cdot b, \qquad b \cdot d = qd \cdot b, \qquad c \cdot d = qd \cdot c \tag{33a}$$

and

$$a \cdot d - qb \cdot c = d \cdot a - q^{-1} b \cdot c = 1 \tag{33b}$$

$$a^* = d, \qquad b^* = -qc, \qquad c^* = -q^{-1} b, \qquad q \in \mathbf{R} \tag{33c}$$

Then, one can write

$$U = \omega^{\mathbf{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} [\mathrm{Ln}(\omega)]^n \mathbf{A}^n \simeq \mathbf{1} + \mathrm{Ln}(\omega)\, \mathbf{A} \tag{34}$$

where **A** is given by equation (23). This expansion was used in Arik and Saracoğlu (1994) to show that the components of the eigenvectors of an $SL_q(2)$ matrix are given by Bessel functions.

Furthermore, one can see the case $N = 2$ treated in Djemai (1995) within this scheme. It results that the conditions (33c) are naturally fulfilled by construction [see equation (34)], with

$$q \equiv \omega = e^{i\pi} = -1 \in \mathrm{R}/q^2 = 1 \ (\neq -1) \tag{35}$$

In this case, the relations (33a) and (33b) become

$$\begin{aligned} &\{a, b\} = \{a, c\} = \{b, d\} = \{c, d\} = 0 \\ &[a, d] = [b, c] = 0, \qquad ad + bc = 1 \end{aligned} \tag{36}$$

Then, from equation (34) one has the following commutation relations:

$$\{a_1, a_3\} = \{a_2, a_3\} = 0, \qquad [a_1, a_2] = 0, \qquad a_1^2 + a_2^2 = a_3^2 \tag{37}$$

4. KNOT FORMALISM AT WORK

In Section 2 we saw how knot theory leads naturally via the Kauffman bracket to the notion of quantum groups. The purpose of this section is to show how a Hopf algebra structure can give rise through its matrix representation to invariants of links, making use of the formalism behind the quantum double construction of Drinfeld (1986) (see also Majid, 1990d).

Let H be a Hopf algebra with basis $\{e_i, i = 0, \ldots, n\}$ and let H^* be a Hopf algebra, dual to H, with generators $\{e^j, j = 0, \ldots, n\}$.

Our quantum double $D(H)$ built on $H^* \otimes H$ is defined such that the Hopf algebras H and H^* are respectively equipped with the following multiplication laws:

$$m(e_i \otimes e_j) = e_i \cdot e_j = A_{ij}^k e_k \tag{38a}$$

$$m^*(e^i \otimes e^j) = e^i \cdot e^j = B_k^{ij} e^k \tag{38b}$$

such that

$$A_{nj}^s B_s^{bq} = \delta_n^b \delta_j^q \tag{39}$$

and where the coproduct Δ for H is the opposite of the multiplication m^* for H^* and the coproduct Δ^* for H^* is the opposite of the multiplication m for H, i.e.,

$$\Delta(e_k) = B_k^{ij} e_i \otimes e_j \Leftrightarrow e_i \otimes e_j = A_{ij}^k \Delta(e_k) \tag{40a}$$

$$\Delta^*(e^k) = A_{ij}^k e^i \otimes e^j \Leftrightarrow e^i \otimes e^j = B_k^{ij} \Delta^*(e^k) \tag{40b}$$

The co-units ϵ for H and ϵ^* for H^* are defined by

$$\epsilon(e_k) = \begin{cases} 1 & \text{if} \quad k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{41a}$$

$$\epsilon^*(e^k) = \begin{cases} 1 & \text{if} \quad k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{41b}$$

Finally, the antipodes γ for H and γ^* for H^* are defined by

$$\gamma(e_k) \equiv e_{-k} \equiv e^k \tag{42a}$$

$$\gamma^*(e^k) \equiv e^{-k} \equiv e_k \tag{42b}$$

such that

$$e_i \cdot e^j = e^j \cdot e_i = \delta_i^j \mathbf{1} \tag{43}$$

The multiplication and the coproduct on our quantum double $D(H)$ are defined respectively by the following relations:

$$\mu\colon \quad D(H) \boxtimes D(H) \to D(H)$$

$$\mu[(e^m \otimes e_n) \boxtimes (e^i \otimes e_j)] = [e^m \otimes e_n] \cdot [e^i \otimes e_j] = \phi_{njr}^{mis} [e^r \otimes e_s] \tag{44a}$$

and

$$\tilde{\Delta}\colon \quad D(H) \to D(H) \boxtimes D(H)$$

$$\tilde{\Delta}(e^i \otimes e_j) = \psi_{jmr}^{ins} (e^m \otimes e_n) \boxtimes (e^r \otimes e_s) \tag{44b}$$

Its dual $D(H)^*$ is also equipped with the following multiplication and coproduct laws:

$$\mu^*\colon \quad D(H)^* \boxtimes D(H)^* \to D(H)^*$$

$$\mu^*[(e_m \otimes e^n) \boxtimes (e_i \otimes e^j)] = [e_m \otimes e^n] \cdot [e_i \otimes e^j] = \psi_{mis}^{njr} [e_r \otimes e^s] \tag{45a}$$

and

$$\tilde{\Delta}^*\colon \quad D(H)^* \to D(H)^* \boxtimes D(H)^*$$

$$\tilde{\Delta}^*(e_i \otimes e^j) = \phi_{ins}^{jmr} (e_m \otimes e^n) \boxtimes (e_r \otimes e^s) \tag{45b}$$

The quantities ϕ and ψ are tied to A and B by the following relations:

$$\phi_{njr}^{mis} = B_r^{mi} A_{nj}^s = \psi_{njr}^{mis} \tag{46a}$$

with

$$\phi_{njr}^{mis} \psi_{aps}^{bqr} = \delta_a^m \delta_p^i \delta_n^b \delta_j^q \tag{46b}$$

The antipode Γ on $D(H)$ may be defined as follows:

$$\begin{aligned}
[\Gamma(e^i \otimes e_j)] \cdot [e^i \otimes e_j] &= [\gamma^*(e^j) \otimes \gamma(e_i)] \cdot [e^i \otimes e_j] \\
&= [\gamma^*(e^j) \cdot e^i] \otimes [\gamma(e_i) \cdot e_j] \\
&= m^*[\gamma^*(e^j) \otimes e^i] \otimes m[\gamma(e_i) \otimes e_j] \\
&= m^*[\gamma^* \otimes \mathrm{Id}](e^j \otimes e^i) \otimes m[\gamma \otimes \mathrm{Id}](e_i \otimes e_j) \\
&= m^*[\gamma^* \otimes \mathrm{Id}]\Delta^*(e^k) \otimes m[\gamma \otimes \mathrm{Id}]\Delta(e_k) \\
&= \epsilon^*(e^k) \otimes \epsilon(e_k) \\
&= \mathbf{1} \otimes \mathbf{1}
\end{aligned} \tag{47}$$

where we have made use of the general definition of an antipode and the relations (39), (40a), (40b), (41a), and (41b).

The relation (47) means that the inverse of any element $e^i \otimes e_j$ of $D(H)$ is given by $\gamma^*(e^j) \otimes \gamma(e_i)$. Similarly, the antipode Γ^* for the dual quantum double $D(H)^*$ is defined such that the inverse of $(e_i \otimes e^j)$ is $\gamma(e_j) \otimes \gamma^*(e^i)$.

In order to render this formalism more clear, we introduce the following diagrammatic notation (see Fig. 23), where we assume that the symbols corresponding to the coefficients A_{ij}^k and B_k^{ij} commute with the e-nodes and

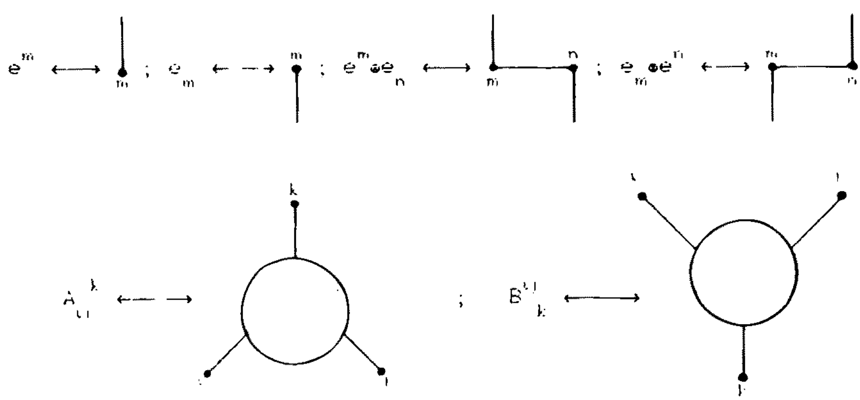

Fig. 23. Diagrammatic representations of the generators e^m, e_m, $e^m \otimes e_n$, $e_m \otimes e^n$, and the coefficients A_{ij}^k and B_k^{ij}.

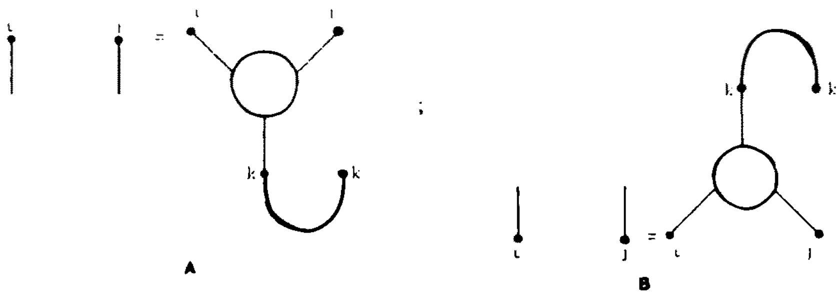

Fig. 24. Diagrammatic representations of the multiplication laws m and m^* [see equations (38a) and (38b)].

that we are summing on repeated indices (lined up). For instance, Figs. 24a,b, 25a,b, and 26 give, respectively, the diagrammatic representations of the relations (38a), (38b); (42a), (42b); and (47). The presence of indices in the diagrams means simply that we are in some matrix representation.

Moreover, we also assume the following relations to be fulfilled:

$$e_i B^{ij}_k A^n_{jm} \cdot e^m = e^m A^n_{mj} B^{ji}_k \cdot e_i$$
$$\Updownarrow \tag{48}$$
$$B^{ij}_k A^n_{jm} e_i \cdot e^m = A^n_{mj} B^{ji}_k e^m \cdot e_i$$

$$(e_m \otimes e^n) \cdot \phi^{mas}_{nbr} \psi^{bqi}_{apj} (e^p \otimes e_q) = (e^p \otimes e_q) \cdot \psi^{qbi}_{paj} \phi^{ams}_{bnr} (e_m \otimes e^n)$$
$$\Updownarrow \tag{49}$$
$$\phi^{mas}_{nbr} \psi^{bqi}_{apj} (e_m \otimes e^n) \cdot (e^p \otimes e_q) = \psi^{qbi}_{paj} \phi^{ams}_{bnr} (e^p \otimes e_q) \cdot (e_m \otimes e^n)$$

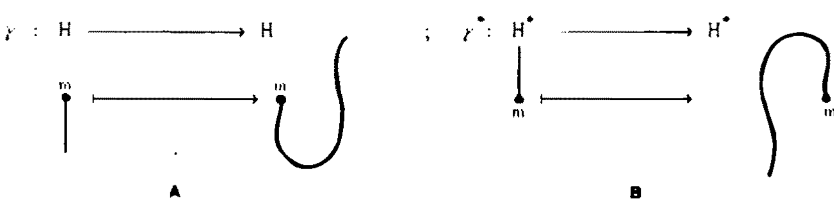

Fig. 25. Diagrammatic representations of antipodes γ and γ^* [see equations (42a), (42b)].

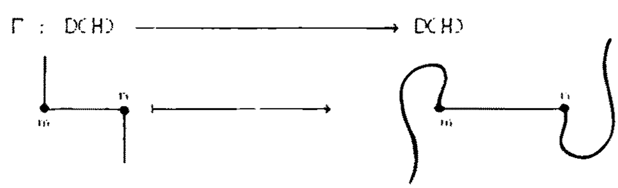

Fig. 26. Diagrammatic representation of the antipode Γ of the quantum double $D(H)$.

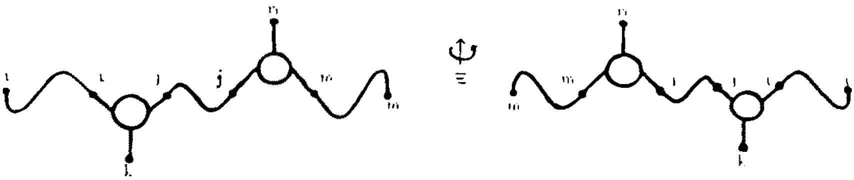

Fig. 27. Diagrammatic representation of the relation (48).

For instance, the relation (48) may be represented by the diagram of Fig. 27.

Let $\mathcal{R} \in D(H) \boxtimes D(H)$ be the element (see Fig. 28)

$$\mathcal{R} = \sum_{m,n} (e^m \otimes e_n) \boxtimes (e_m \otimes e^n) \tag{50}$$

with the following identities $\in (D(H) \boxtimes D(H) \boxtimes D(H)$:

$$\mathcal{R}_{12} = \sum_{m,n} (e^m \otimes e_n) \boxtimes (e_m \otimes e^n) \boxtimes (\mathbf{1} \otimes \mathbf{1}) \tag{51a}$$

$$\mathcal{R}_{13} = \sum_{a,b} (e^a \otimes e_b) \boxtimes (\mathbf{1} \otimes \mathbf{1}) \boxtimes (e_a \otimes e^b) \tag{51b}$$

$$\mathcal{R}_{23} = \sum_{p,q} (\mathbf{1} \otimes \mathbf{1}) \boxtimes (e^p \otimes e_q) \boxtimes (e_p \otimes e^q) \tag{51c}$$

Then, using the relations (44a), (45a), and (49), we can easily prove that this element obeys the following quantum Yang–Baxter equation:

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{52}$$

Then, $\mathcal{R}$ denotes an algebraic solution to the Yang–Baxter equation, which can be interpreted as the knot-theoretic $\mathcal{R}$-matrix in some representation (see Section 2).

As we know from Section 2, to ensure the existence of a link invariant the knot theory requires some relations between $\mathcal{R}$, $\mathcal{R}^{-1}$ and the creation and annihilation matrices [see equations (5a)–(5d)].

Since any quantum double is *quasitriangular*, then $\mathcal{R}^{-1}$ is defined by the relation

$$(\Gamma \otimes \mathrm{Id})(\mathcal{R}) = \mathcal{R}^{-1} \tag{53}$$

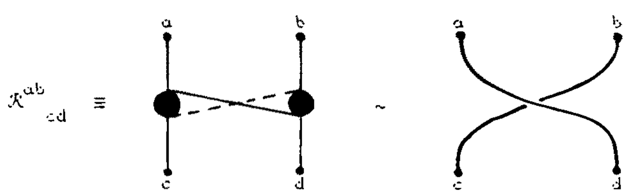

Fig. 28. Diagrammatic representation of the $\mathcal{R}$-matrix.

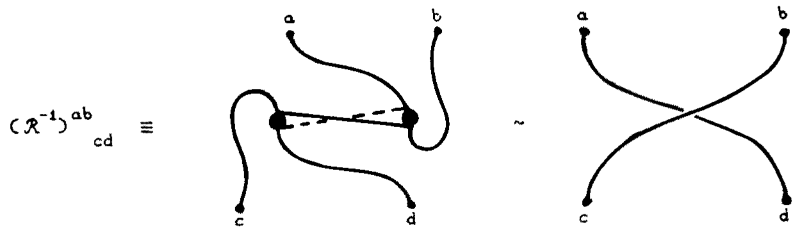

Fig. 29. Diagrammatic representation of $\mathcal{R}^{-1}$.

It is represented by the diagram of Fig. 29.

It follows that the relations (5a)–(5d) are diagrammatically represented by the same diagrams as in Figs. 15a–15d, where the creation and annihilation matrices are given by the same diagrams as in Fig. 13. For instance, the relation (5c) may be reproduced as shown in Fig. 30.

Finally, we remark that our twist conditions required by knot theory are intimately tied to the quantum double Hopf algebra structure for the quantum group.

This quantum double construction can be applied for the case of our Hopf algebra H equipped with a basis consisting of Schwinger matrices. The roles of the above basis $\{e_i\}$ and $\{e^i\}$ are then played respectively by $\{\gamma_m\}$ and $\{\gamma^m\}$, and one can follow the same procedure as above.

Now we show how all this formalism can be seen from a quantum mechanical point of view. We will use the same notations as in Djemai (1996).

First, consider the case of a quantum particle moving in a two-dimensional phase space. The operator $\gamma(a, b)$, which gives rise to the translations

$$q \to q + a, \qquad p \to p + b \tag{54}$$

can be obtained by the following correspondence:

$$\gamma_{mn} = \omega^{mn/2}\mathbf{U}^m \cdot \mathbf{V}^n \to \gamma(a, b) = e^{i(a\mathbf{p} - b\mathbf{q})} \tag{55}$$

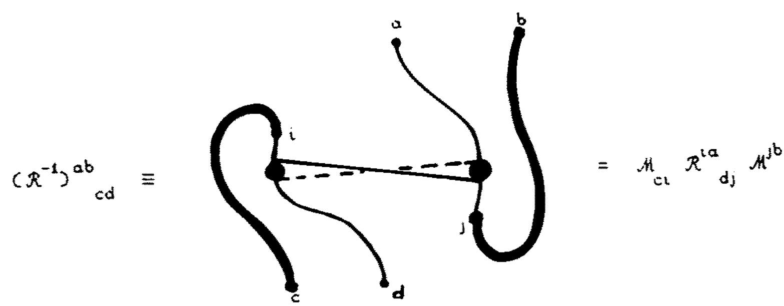

Fig. 30. Diagrammatic representation of the relation (5c).

where a and b are real parameters characterizing the translations along the position and momentum directions, respectively, and where the Hermitian operators $\mathbf{q}$ and $\mathbf{p}$ generate (with $\mathbf{1}$) the fundamental Heisenberg algebra:

$$[\mathbf{q}, \mathbf{p}] = i\hbar\mathbf{1} \tag{56}$$

The algebra H generated by $\gamma(a, b)$ is of "Hopf" type. It is equipped with a "multiplication" m, a "coproduct" Δ, a "co-unit" ϵ, and an "antipode" Υ defined, respectively, by

$$\begin{aligned} m[\gamma(a, b) \otimes \gamma(c, d)] &= \gamma(a, b)\cdot\gamma(c, d) \\ &= e^{-i\alpha_2[(a,b);(c,d)]}\delta(a + c - r)\delta(b + d - s)\gamma(r, s) \end{aligned} \tag{57}$$

$$\Delta[\gamma(r, s)] = e^{i\alpha_2[(a,b);(c,d)]}\delta(a + c - r)\delta(b + d - s)\gamma(a, b) \otimes \gamma(c, d) \tag{58}$$

$$\epsilon[\gamma(a, b)] = \begin{cases} \mathbf{1} & \text{if} \quad a = b = 0 \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{59}$$

$$\Upsilon[\gamma(a, b)] = \gamma(-a, -b) \tag{60}$$

Similarly, the "Hopf" algebra H^* "dual" to H is generated by the elements $\gamma^+(a, b)$ which are obtained from the following correspondence:

$$\gamma^{mn} = \gamma^+_{mn} = \gamma_{-m,-n} \rightarrow \gamma^+(a, b) = e^{-i(a\mathbf{p}-b\mathbf{q})} \tag{61}$$

H^* is endowed with a "multiplication" m^*, a "coproduct" Δ^*, a "co-unit" ϵ^*, and an "antipode" Υ^* defined, respectively, by

$$\begin{aligned} m^*[\gamma^+(a, b) \otimes \gamma^+(c, d)] &= \gamma^+(a, b)\cdot\gamma^+(c, d) \\ &= e^{-i\alpha_2[(a,b);(c,d)]}\delta(a + c - r)\delta(b + d - s)\gamma^+(r, s) \end{aligned} \tag{62}$$

$$\Delta^*[\gamma^+(r, s)] = e^{i\alpha_2[(a,b);(c,d)]}\delta(a + c - r)\delta(b + d - s)\gamma^+(a, b) \otimes \gamma^+(c, d) \tag{63}$$

$$\epsilon^*[\gamma^+(a, b)] = \begin{cases} \mathbf{1} & \text{if} \quad a = b = 0 \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{64}$$

$$\Upsilon^*[\gamma^+(a, b)] = \gamma^+(-a, -b) \tag{65}$$

The "Hopf" algebra H^* can be considered as describing the evolution of the *anti*-partner of the above quantum particle in the phase space. This interpretation follows from the fact that

$$\gamma^+(a, b) = \gamma(-a, -b) \tag{66}$$

which physically means that the antiparticle translates in the *opposite* direction of that of the associated particle.

In the above relations, the *tensor product* $\otimes$ may be interpreted as being the *association* of two particles (or two antiparticles).

Furthermore, it is straightforward to think of the physical situation described by the quantum double structure $D(H)$ as nothing else than the evolution of a particle–antiparticle pair in the quantum phase space. Concretely, the dual $D(H)^*$ will describe the same physical situation.

The "Hopf" algebra $D(H)$ whose elements are of the form $\gamma^+(a, b) \otimes \gamma(c, d)$ is equipped with a "multiplication" μ, a "coproduct" $\tilde{\Delta}$, a "co-unit" $\tilde{\epsilon}$, and an antipode Γ defined, respectively, by

$$\begin{aligned}
&\mu\{[\gamma^+(a, b) \otimes \gamma(c, d)] \boxtimes [\gamma^+(a', b') \otimes \gamma(c', d')]\} \\
&\quad = [\gamma^+(a, b) \otimes \gamma(c, d)] \cdot [\gamma^+(a', b') \otimes \gamma(c', d')] \\
&\quad = e^{-i\alpha 2[(a,b);(a',b')]} e^{-i\alpha 2[(c,d);(c',d')]} \delta(a + a' - e)\delta(b + b' - f) \\
&\qquad \times \delta(c + c' - g)\delta(d + d' - h)\gamma^+(e, f) \otimes \gamma(g, h) \qquad (67)
\end{aligned}$$

$$\begin{aligned}
&\tilde{\Delta}[\gamma^+(e, f) \otimes \gamma(g, h)] \\
&\quad = e^{i\alpha 2[(a,b);(a',b')]} e^{i\alpha 2[(c,d);(c',d')]} \\
&\qquad \times \delta(a + a' - e)\delta(b + b' - f)\delta(c + c' - g)\delta(d + d' - h) \qquad (68) \\
&\qquad \times [\gamma^+(a, b) \otimes \gamma(c, d)] \boxtimes [\gamma^+(a', b') \otimes \gamma(c', d')]
\end{aligned}$$

$$\tilde{\epsilon}[\gamma^+(a, b) \otimes \gamma(c, d)] = \begin{cases} \mathbf{1} \otimes \mathbf{1} & \text{if} \quad a = b = c = d = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (69)$$

$$\begin{aligned}
\Gamma[\gamma^+(a, b) \otimes \gamma(c, d)] &= \Upsilon^*[\gamma^+(a, b)] \otimes \Upsilon[\gamma(c, d)] \\
&= \gamma^+(-a, -b) \otimes \gamma(-c, -d) \qquad (70)
\end{aligned}$$

Finally, we conclude by remarking that the $\mathcal{R}$-matrix $\in D(H) \boxtimes D(H)$ may be interpreted as describing an *interaction* between two pairs of particle–antiparticle, in analogy with the usual Feynman diagrams (Feynman, 1964).

5. CONCLUSION

In previous work (Djemai, 1996) we described quantum mechanics as a *noncommutative symplectic geometry* using the Weyl–Wigner–Moyal quantization technique, the Weyl–Schwinger realization of the Heisenberg group, and the matrix differential geometry associated with the matrix algebra $M_N(\mathbb{C})$ generated by Schwinger matrices.

The *matrix (quantum) Hamiltonian formalism* developed in Djemai (1996) was used in Djemai (1995) to show the presence of a *braiding* and then of a *Yang–Baxter algebra structure* behind our *lattice quantum phase* *space*. In fact, from this braiding we obtained a nontrivial $\mathcal{R}$-matrix solution for arbitrary N. Moreover, a second nontrivial $\mathcal{R}$-matrix solution was presented following the method used in Dubois-Violette and Launer (1990). We also discussed in Djemai (1995) the construction of the *quantum group* associated with the first $\mathcal{R}$-matrix solution for the case $N = 2$.

In the present work, we have completed the latter discussion and showed that the case $N = 2$ is somewhat trivial; it would be more interesting to investigate the cases $N \geq 3$. In addition to the case $N = 3$, we plan to study the construction of the quantum group associated with the second $\mathcal{R}$-matrix in a future work. The discussion of the case $N = 2$ was continued by deriving the commutation relations following from the identification of infinitesimal generators of the $SU_q(2)$ with our quantum (operator) algebra following the arguments presented in Vokos *et al.* (1990), Corrigan *et al.* (1990), and Arik and Saracoğlu (1994).

The second main purpose of this work was to show how to go from a knot theory to a quantum group structure and back. For this purpose, we used the Drinfeld's (1986) quantum double, which is one of the most important quantum group constructions and plays an important role in physics (Podles and Woronowicz, 1990; Brzezinski and Majid, 1993; Majid, 1992). The quantum double constructed here is based on our Hopf algebra generated by Schwinger matrices.

In order to show that this quantum double construction leads naturally to knot theory, it has been necessary to introduce a new diagrammatic notation. Our approach appears to be more general than the one proposed by Kauffman (1990a). Nevertheless, we think that this question still requires further investigation.

In other respects, our approach also enables us to give a physical interpretation of this quantum double by describing the matrix quantum mechanics of a particle–antiparticle system moving in our lattice quantum phase space. The continuous version was also studied.

Finally, we plan to complete this work by presenting other descriptions of quantum mechanics, its q-deformation, its Lagrangian and Newtonian formulations and related subjects.

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